Quiz 2

Date: September 26, 2025

Question 1. 5 points

Identify the error in the following proof, and explain why it is incorrect.

Proposition: Let G be a $2^n \times 2^n$ checkerboard, and let $G - \square$ be the result of removing one square from G. Then, $G - \square$ is L-tileable (i.e., $G - \square$ can be covered by an L-shape made up of three squares).
Proof: The total number of squares in $G - \square$ are $2^n \cdot 2^n - 1 = 4^n - 1$. In class we proved that $3 \mid 4^n - 1$, and so $4^n - 1 = 3k$ for some integer k. So, as an L-shaped block has 3 squares, this means we can L-tile $G - \square$ with k such L-shaped blocks.

Solution: For a shape S, just because the number of squares in S is a multiiple of the number of squares in L does NOT mean that S is L-tileable. For example, if S is a single row of 3-squares, then S is not L-tilable.

Rubric:

- **3 points:** identifying the issue,
- **2 points:** giving a counterexample.

Question 2. 10 points

Let $n \geqslant 1$ be an integer, and let G be a $2 \times n$ checkerboard. Prove that the number of ways of tiling G with a 2×1 domino is F_{n+1}.

Solution: We proceed by strong induction.

<u>**Base cases:**</u> When $n = 1$ there is only $1 = F_2$ ways to tile a 2×1 checkerboard with 2×1 dominoes. For $n = 2$ there are clearly only $2 = F_3$ ways, given by either having the dominos both be vertical or both be horizontal.

<u>**Strong induction hypothesis:**</u> Assume that the equality holds for all $m < n$, and that $n \geqslant 3$. Note that for the first column, there are two cases:

<u>**Case 1:**</u> The first column is tiled by a single domino. In this case, G', the result of removing the first column from G, is a $2 \times n$ checkerboard and the number of tilings of G and G' are the same. So, by the strong induction hypothesis there are F_{n+1} ways to tile this with a

2×1 domino.

<u>Case 2:</u> The first column is not taken up by a single domino. In this case it must be the case that the first two columns are taken up by two horizontally-placed 2×1 dominoes. So, G', the result of removing the first two columns from G, is a $2 \times (n-2)$ checkerboard whose number of tilings is the same as that of G. By induction hypothesis there are F_n of these.

So, as Case 1 and Case 2 account for all possibilities and are disjoint there are a total of $F_{n+1} + F_n = F_{n+2}$ number of tilings, as desired.

Rubric:

- **1 points:** recognizing that one should use strong induction,
- **2 points:** Correctly identifying the two appropriate base cases, and verifying them.
- **3 points:** correctly arguing in one case,
- **3 points:** correctly arguing in the second case,
- **1 points:** correctly combining the two cases to deduce the strong inductive step.